JOURNAL OF COMPUTATIONAL PHYSICS **25**, 32–55 (1977)

Integration Formulas for the Wave Equation in n Space Dimensions*

A. H. STROUD

Department of Mathematics, Texas A & M University, College Station, Texas 77843

Received August 30, 1976; revised December 7, 1976

We consider the wave equation in n space dimensions $(\partial^2 u/\partial t^2) - (\partial^2 u/\partial x_1^2) - \cdots - (\partial^2 u/\partial x_n^2) = F(x_1, \ldots, x_n, t)$. We derive formulas to approximate u at a point $(x_{01}, \ldots, x_{0n}, t_0)$ assuming $u(x_1, \ldots, x_n, 0) = f(x_1, \ldots, x_n)$ and $u_t(x_1, \ldots, x_n, 0) = g(x_1, \ldots, x_n)$ are given. The formulas are exact when f, g, and F are arbitrary polynomials of degree $\leqslant d$, for various integers d, and are approximations to integrals which represent the solution.

1. INTRODUCTION

We consider the problem of solving the wave equation in n space dimensions for $n \geqslant 2$. This is the problem of finding $u(x, t) \equiv u(x_1, \ldots, x_n, t)$ to satisty

$$u_{tt} - \Delta_n u = F(x, t), \qquad u(x, 0) = f(x), \qquad u_t(x, 0) = g(x) \tag{1.1}$$

Here $F(x, t)$, $f(x)$, $g(x)$, where $x = (x_1, \ldots, x_n)$, are given functions and

$$\Delta_n \equiv \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2} \qquad u_t \equiv \frac{\partial u}{\partial t} \qquad u_{tt} \equiv \frac{\partial^2 u}{\partial t^2}.$$

If $\phi(x, t) \equiv \phi(f; x, t)$ is the solution of

$$\phi_{tt} - \Delta_n \phi = 0, \qquad \phi(x, 0) = f(x), \qquad \phi_t(x, 0) = 0; \tag{1.2}$$

if $\psi(x, t) \equiv \psi(g; x, t)$ is the solution of

$$\psi_{tt} - \Delta_n \psi = 0, \qquad \psi(x, 0) = 0, \qquad \psi_t(x, 0) = g(x); \tag{1.3}$$

and if $\zeta(x, t) \equiv \zeta(F; x, t)$ is the solution of

$$\zeta_{tt} - \Delta_n \zeta = F(x, t), \qquad \zeta(x, 0) = 0, \qquad \zeta_t(x, 0) = 0; \tag{1.4}$$

then the solution of (1.1) is $u = \phi + \psi + \zeta$.

The usual numerical techniques for solving (1.1) obtain the solution on a mesh of points. (See, for example, Forsythe and Wasow [7, Chap. 4] or Mitchell [8, Chap. 5].) If a solution is required at a large number of points then one of those methods will

* This work was supported by NSF Grant DCR75-04545.

ISSN 0021-9991

probably be the most efficient. However, in some cases the solution may be desired at only *one point* (or a small number of points). It may then be desirable to have a method especially designed for such a problem. Such a method would also be useful for spot checking the solution obtained on a mesh. The methods to be described below are of this type.

For $n = 1$ the solution of (1.1) is the d'Alembert solution

$$u(x, t) = \tfrac{1}{2}[f(x-t) + f(x+t)] + \tfrac{1}{2}\int_{x-t}^{x+t} g(\chi)\, d\chi + \tfrac{1}{2}\int_0^t \int_{x-t+\tau}^{x+t-\tau} F(\chi, \tau)\, d\chi\, d\tau \tag{1.5}$$

(see, e.g. Weinberger [6, p. 26]). In this case one can approximate u at a point by using a quadrature formula for the interval $x - t \leqslant \chi \leqslant x + t$ and a cubature formula for the triangle

$$0 \leqslant \tau \leqslant t, \qquad x - t + \tau \leqslant \chi \leqslant x + t - \tau.$$

(Cubature formulas for triangles are given by Stroud [4] and by Lyness and Jesperson [3].)

For $n \geqslant 2$ the solution of problem (1.1) is also known in terms of integrals. Courant and Hilbert [1, p. 682] show that the solution of (1.3) can be written as

$$\psi(g; x, t) = \frac{t}{\Gamma(n/2)}\left(\frac{\partial}{\partial t^2}\right)^{(n-2)/2} [t^{n-3}\mathscr{H}(x, t)] \qquad \text{for } n \text{ even}, \tag{1.6}$$

$$\psi(g; x, t) = \frac{\pi^{1/2}}{2\Gamma(n/2)}\left(\frac{\partial}{\partial t^2}\right)^{(n-3)/2} [t^{n-2}\mathscr{Q}(x, t)] \qquad \text{for } n \text{ odd}, \tag{1.7}$$

where

$$\partial/\partial t^2 = (1/2t)(\partial/\partial t),$$

$$\mathscr{H}(x, t) = \int_0^t \frac{r\mathscr{Q}(x, r)}{(t^2 - r^2)^{1/2}}\, dr, \tag{1.8}$$

$$\mathscr{Q}(x, t) = \frac{1}{\omega_n} \underset{|\beta|=1}{\int\cdots\int} g(x_1 + \beta_1 t, \ldots, x_n + \beta_n t)\, d\beta. \tag{1.9}$$

The integral in (1.9) is over the surface of the unit n-sphere defined by

$$|\beta|^2 \equiv \beta_1^2 + \beta_2^2 + \cdots + \beta_n^2 = 1.$$

The "surface area" of this sphere is $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$.

The solution of (1.2) can be written as

$$\phi(f; x, t) = (\partial/\partial t)\, \psi(f; x, t). \tag{1.10}$$

This fact is a special case of Stokes' rule [6, p 370]. The solution of (1.4) can be written as

$$\zeta(F; x, t) = \int_0^t \psi(F; x, t - \tau; \tau)\, d\tau \tag{1.11}$$

where $\psi(F; x, t - \tau; \tau)$ denotes the solution of

$$\psi_{tt} - \Delta_n \psi = 0, \qquad \psi(x, 0; \tau) = 0, \qquad \psi_t(x, 0; \tau) = F(x, \tau).$$

Representation (1.11) is a special case of Duhamel's principle [1, p. 203].

The purpose of the present paper is to use the above representations of ψ, ϕ, and ζ to derive formulas for approximating these functions at a point. The formulas will be exact when f, g, and F are arbitrary polynomials of degree $\leqslant d$, for specified integers d. Because of the differentiation operator in (1.6) and (1.7) and the occurrence of t in the argument of g in (1.8) and (1.9), the evaluation of ϕ for $n \geqslant 2$ (and ψ for $n \geqslant 4$) requires a cubature whose integrand involves partial derivatives of f (respectively g). In some situations these derivatives may be readily available and in other situations they may not. Consequently we will derive two classes of formulas. The simpler class is based on direct cubature of a function $\hat{f}$ (respectively $\hat{g}$) which is constructed by the user from f (respectively g) and its derivatives. The theory for these is given in Sections 4 and 5; application of these formulas for ψ to the computation of ζ is discussed in Section 8. The reader interested only on these formulas may omit Sections 2, 3, 6, and 7 entirely.

A second class of formulas (which are not strictly cubature formulas) are based on the theory of Sections 2 and 3 and are presented in Sections 6 and 7. These approximate ψ and ϕ in terms of g and f only. Formulas of this type are not explicitly discussed for ζ but results analogous to those given in Section 8 could also be obtained.

Examples are given in Section 9. These examples were computed using subroutines listed in report [5]. This report contains eleven subroutines which incorporate the formulas derived below.

2. Notation

We introduce some additional notation.

(i) $i = -1^{1/2}$.

(ii) If v is a real number, $[v]$ denotes the largest integer $\leqslant v$.

(iii) $d, m, s, \alpha_1, \ldots, \alpha_n, \nu, \sigma$ are nonnegative integers.

(iv) $H_{m,n} \equiv H_{m,n}(x_1, \ldots, x_n)$ denotes a homogeneous polynomial of degree m in the n variables; that is

$$H_{m,n}(\alpha x_1, \ldots, \alpha x_n) \equiv \alpha^m H_{m,n}(x_1, \ldots, x_n) \qquad \text{for all } \alpha.$$

(v) A polynomial $Q(x_1, \ldots, x_n)$ is called a harmonic polynomial if $\Delta_n Q \equiv 0$.

(vi) $H^h_{m,n}$ denotes a homogeneous harmonic polynomial of degree m in n variables.

(vii) $\Pi_{m,n}$ denotes a basis for the vector space of all $H_{m,n}$. One such basis is the set of all monomials

$$x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \qquad \alpha_1 + \alpha_2 + \cdots + \alpha_n = m,$$

(viii) The number of elements in a basis $\Pi_{m,n}$ will be denoted by $\gamma(m, n)$. It is well known that

$$\gamma(m, n) = (m + n - 1)!/m!(n - 1)!.$$

(ix) $\Pi^h_{m,n}$ denotes a basis for the vector space of all $H^h_{m,n}$. $\Pi^h_{0,n}$ contains one element: $H^h_{0,n} \equiv 1$. For $m > 0$, $\Pi^h_{m,2}$ consists of two elements which can be taken as Re z^m, Im z^m, $z = x + iy$. Below we will mention how a basis $\Pi^h_{m,n}$, $n > 2$, can be found.

(x) The number of elements in a basis $\Pi^h_{m,n}$ will be denoted by $\delta(m, n)$. It is known that

$$\begin{aligned} \delta(m, n) &= \gamma(0, n) = 1 && \text{for } m = 0, \\ &= \gamma(1, n) = n && \text{for } m = 1, \\ &= \gamma(m, n) - \gamma(m - 2, n) && \text{for } m \geqslant 2. \end{aligned} \tag{2.1}$$

(See [2, p. 237].)

(xi) $C_m{}^\lambda(\tau)$ denotes the mth degree Gegenbauer polynomial of order λ where $\lambda > -\frac{1}{2}$. $C_m{}^\lambda(\tau)$ is the mth degree polynomial which is orthogonal on the interval $-1 \leqslant \tau \leqslant +1$, with respect to the weight function $(1 - \tau^2)^{\lambda-1/2}$ for all polynomials of degree $<m$. We assume the usual standardization for $C_m{}^\lambda$, namely

$$C_m{}^\lambda(1) = (m + 2\lambda - 1)!/m!(2\lambda - 1)!$$

It is well known that $C_m{}^\lambda(\tau)$ is an even function of τ for m even an odd function for m odd. We will denote the coefficient of τ^k in $C_m{}^\lambda(\tau)$ by $e^\lambda_{m,k}$. That is

$$C_m{}^\lambda(\tau) = e^\lambda_{m,m}\tau^m + e^\lambda_{m,m-2}\tau^{m-2} + \cdots.$$

(xii) A polynomial $Q(x_1, \ldots, x_n, t)$ is called a wave polynomial if $\Delta_n Q \equiv Q_{tt}$.

(xiii) $H^w_{m,n+1} \equiv H^w_{m,n+1}(x_1, \ldots, x_n, t)$ denotes a homogeneous wave polynomial of degree m in n space variables,

(xiv) $\Pi^w_{m,n+1}$ denotes a basis for the vector space of all $H^w_{m,n+1}$. In Section 3 we will see that wave polynomials are closely related to harmonic polynomials.

(xv) By U_n we mean the surface of the unit n-sphere, defined in $(\beta_1, \ldots, \beta_n)$-space by

$$\beta_1{}^2 + \cdots + \beta_1{}^2 = 1.$$

(xvi) By a cubature formula of degree d for U_n we mean a set of points and coefficients

$$(\beta_{k1},\dots,\beta_{kn})\ B_k, \qquad k=1,\dots,K \tag{2.2}$$

with the property that the approximation

$$(1/\omega_n)\int_{U_n}\cdots\int g(\beta_1,\dots,\beta_n)\,d\beta \simeq \sum_{k=1}^{K} B_k g(\beta_{k1},\dots,\beta_{kn})$$

is exact when g is an arbitrary polynomial of degree $\leqslant d$, and d is the largest integer for which this is true. We note that

$$\sum_{k=1}^{K} B_k = 1$$

and we always assume

$$\beta_{k1}^2+\cdots+\beta_{kn}^2=1, \qquad k=1,\dots,K.$$

For $n=2$ a formula of degree d for U_2, the circumference of the unit circle, consists of $K=d+1$ points equally spaced around U_2 with $B_1=\cdots=B_K=1/K$. Stroud [4] gives a variety of formulas for U_n, $n\geqslant 3$. In the computer programs to be described we use Gauss product formulas; such a formula of degree $d=2d_0-1$ contains $K=2d_0^{n-1}$ points.

If $u(x_1,\dots,x_n)$ satisfies $\Delta u_n=0$ then

$$u(0,\dots,0)=(1/\omega_n)\int_{U_n}\cdots\int u(\beta_1,\dots,\beta_n)\,d\beta.$$

This is a special case of Poisson's integral formula (see, e.g., Courant and Hilbert [1, p. 265]). It follows that if (2.2) is a cubature formula of degree d for U_n then

$$H_{m,n}^h(0,\dots,0)=\sum_{k=1}^{K} B_k H_{m,n}^h(\beta_{k1},\dots,\beta_{kn}) \tag{2.3}$$

for $m\leqslant d$.

3. Harmonic Polynomials and Wave Polynomials

We will use two lemmas about harmonic polynomials.

Lemma 1. *Define* $\tilde\rho=(t^2+\tilde x_1{}^2+\cdots+\tilde x_n{}^2)^{1/2}$. *Let* $H_{s,n}^h(\tilde x_1,\dots,\tilde x_n)$ *be any homogeneous harmonic polynomial of degree* s *in* $\tilde x_1,\dots,\tilde x_n$ *and let* $s\leqslant m$. *Then*

$$\tilde Q_{m,s,n}(\tilde x_1,\dots,\tilde x_n t)\equiv\tilde\rho^{m-s}C_{m-s}^{s+(n-1)/2}(t/\tilde\rho)\,H_{s,n}^h(\tilde x_1,\dots,\tilde x_n) \tag{3.1}$$

is a $H_{m,n+1}^h(\tilde x_1,\dots,\tilde x_n,t)$.

LEMMA 2. *Let $\tilde{Z}_{m,s,n}$ denote the set of all $\tilde{Q}_{m,s,n}$ defined by* (3.1) *as $H^h_{s,n}$ varies over a basis $\Pi^h_{s,n}$. Then the union of the sets*

$$\tilde{Z}_{m,0,n}\,,\ \tilde{Z}_{m,1,n}\,,\ldots,\ \tilde{Z}_{m,m,n}$$

consistents of $\delta(m, n+1)$ linearly independent polynomials which form a basis $\Pi^h_{m,n+1}$. Furthermore

$$\delta(m, n+1) = \gamma(m-1, n) + \gamma(m, n), \qquad m \geqslant 1.$$

Proofs of the statements in these lemmas are given by Erdélyi [2, p. 237–239]. These properties of harmonic polynomials imply analogous properties of wave polynomials which we now give.

LEMMA 3. *Define $\rho = (t^2 - x_1^2 - \cdots - x_n^2)^{1/2}$ and let $s \leqslant m$. Then*

$$Q_{m,s,n}(x_1,\ldots, x_n, t) \equiv \rho^{m-s} C^{s+(n-1)/2}_{m-s}(t/\rho)\, H^h_{s,n}(x_1,\ldots, x_n) \tag{3.2}$$

is a $H^w_{m,n+1}(x_1,\ldots, x_n, t)$.

Proof. Since C^λ_{m-s} is either an even or odd function we have

$$\rho^{m-s} C^\lambda_{m-s}(t/\rho) = e^\lambda_{m-s,m-s} t^{m-s} + e^\lambda_{m-s,m-s-2} t^{m-s-2}\rho^2 + \cdots.$$

Therefore only even powers of ρ enter into (3.2). Define $\tilde{x}_k = ix_k$, $k = 1,\ldots, n$. Then $\tilde{\rho} = \rho$ and

$$\tilde{Q}_{m,s,n}(\tilde{x}_1,\ldots, \tilde{x}_n, t) = i^s Q_{m,s,n}(x_1,\ldots, x_n, t).$$

Therefore

$$\frac{\partial^2 \tilde{Q}_{m,s,n}}{\partial t^2} = i^s \frac{\partial^2 Q_{m,s,n}}{\partial t^2}, \quad \frac{\partial^2 \tilde{Q}_{m,s,n}}{\partial \tilde{x}_k^2} = -i^s \frac{\partial^2 Q_{m,s,n}}{\partial x_k^2}, \qquad k = 1,\ldots, n.$$

It follows that $Q_{m,s,n}$ is a $H^w_{m,n+1}$.

THEOREM 1. *For fixed m let $Z_{m,s,n}$ denote the set of all $Q_{m,s,n}$ defined by* (3.2) *as $H^h_{s,n}$ varies over a basis $\Pi^h_{s,n}$. Define*

$Z^f_{m,n}$ = union of all $Z_{m,s,n}$ *for $m - s$ even, $0 \leqslant s \leqslant m$;*

$Z^g_{m,n}$ = *union of all $Z_{m,s,n}$ for $m - s$ odd, $0 \leqslant s \leqslant m$.*

Then the following statements are true.

(i) *The union of the sets $Z^f_{m,n}$ and $Z^g_{m,n}$ consists of $\delta(m, n+1) = \gamma(m-1, n) + \gamma(m, n)$ linearly independent polynomials which form a basis $\Pi^w_{m,n+1}$.*

(ii) *$Z^f_{m,n}$ consists of $\gamma(m, n)$ polynomials $H^w_{m,n+1}$, each of which satisfies*

$$(\partial/\partial t)\, H^w_{m,n+1}(x_1,\ldots, x_n, t)|_{t=0} = 0. \tag{3.3}$$

Furthermore the set of polynomials

$$F_{m,n} \equiv \{H^w_{m,n+1}(x_1, \ldots, x_n, 0) : H^w_{m,n+1} \text{ is in } Z^f_{m,n}\}$$

is a basis $\Pi_{m,n}$. *Therefore if* $f(x_1, \ldots, x_n)$ *is any* $H_{m,n}$ *the solution of* (1.2) *is a linear combination of the polynomials in* $Z^f_{m,n}$.

(iii) $Z^g_{m,n}$ *consists of* $\gamma(m-1, n)$ *polynomials* $H^w_{m,n+1}$, *each of which satisfies*

$$H^w_{m,n+1}(x_1, \ldots, x_n, 0) = 0. \tag{3.4}$$

Furthermore the set of polynomials

$$G_{m-1,n} \equiv \left\{\frac{\partial}{\partial t} H^w_{m,n+1}(x_1, \ldots, x_n, t)|_{t+0} : H^w_{m,n+1} \text{ is in } Z^g_{m,n}\right\}$$

is a basis $\Pi_{m-1,n}$. *Therefore if* $g(x_1, \ldots, x_n)$ *is any* $H_{m-1,n}$ *the solution of* (1.3) *is a linear combination of the polynomials in* $Z^g_{m,n}$.

Proof. The truth of statement (i) follows from Lemma 2 and the correspondence between harmonic polynomials and wave polynomials implied by Lemmas 1 and 3.

Next consider (ii). Since $Z_{m,s,n}$ contains $\delta(s, n)$ polynomials the number of polynomials in $Z^f_{m,n}$ is

$$\begin{aligned} &\delta(m, n) + \delta(m-2, n) + \cdots + \delta(2, n) + \delta(0, n) && \text{for } m \text{ even,} \\ &\delta(m, n) + \delta(m-2, n) + \cdots + \delta(3, n) + \delta(1, n) && \text{for } m \text{ odd.} \end{aligned}$$

By (2.1) each of these sums equals $\gamma(m, n)$.

Now we show that (3.3) is valid. Each of the $H^w_{m,n+1}$ in $Z^f_{m,n}$ can be written as (3.2) for some $H^h_{s,n}$ with $m - s$ even. Then $H^w_{m,n+1}$ contains only even powers of t and then it follows that (3.3) is true.

Now we show that $F_{m,n}$ is a basis $\Pi_{m,n}$. Since $F_{m,n}$ contains the same number of elements as a basis we must only show that the polynomials in $F_{m,n}$ are linearly independent. If these polynomials were linearly dependent there would exist a wave polynomial $Q(x, t) \not\equiv 0$ for which

$$Q(x, 0) = 0, \qquad (\partial/\partial t)\, Q(x, t)|_{t=0} = 0.$$

This is impossible because the only solution of (1.2) with $f(x) \equiv 0$ is $u(x, t) \equiv 0$.

Next consider (iii). By part (i) and the first statement of part (ii) it follows that $Z^g_{m,n}$ consists of $\gamma(m-1, n)$ polynomials. Since $m - s$ is odd, each $H^w_{m,n+1}$ in $Z^g_{m,n}$ can be written as

$$\begin{aligned} H^w_{m,n+1} &= [e^\lambda_{m-s,m-s} t^{m-s} + \cdots + e^\lambda_{m-s,1} t \rho^{m-s-1}]\, H^h_{s,n}, \\ \lambda &= s + (n-1)/2, \end{aligned}$$

for some $H_{s,n}^h$. This implies (3.4) and that the polynomials in $G_{m-1,n}$ have the form

$$(\partial/\partial t)\, H_{m,n+1}^w \,|_{t=0} = (-1)^{(m-s-1)/2}\, e_{m-s,1}^{\lambda}(x_1^2 + \cdots + x_n^2)^{(m-s-1)/2}\, H_{s,n}^h\,. \tag{3.5}$$

The proof that the polynomials in $G_{m-1,n}$ are linearly independent is analogous to the proof that the polynomials in $F_{m,n}$ are linearly independent. This completes the proof of Theorem 1.

4. Cubature Formulas for ψ

4.1. $n = 2\mu$

We will write $t^{n-3}\mathscr{H}(x, t)$ in a form in which the differentiation with respect to t in (1.6) can be carried out. From (1.8) and (1.9)

$$t^{n-3}\mathscr{H}(x, t) = \frac{t^{n-3}}{\omega_n} \int_0^t \frac{r}{(t^2 - r^2)^{1/2}} \int\cdots\int_{U_n} g(x_1 + \beta_1 r, \ldots, x_n + \beta_n r)\, d\beta\, dr. \tag{4.1}$$

In this integral we make the change of variables

$$\xi_l = r\beta_l\,, \qquad l = 1, \ldots, n.$$

Then

$$r^{n-1}\, d\beta\, dr = d\xi_1\, d\xi_2 \cdots d\xi_n \equiv d\xi,$$
$$|\,\xi\,|^2 \equiv \xi_1^2 + \cdots + \xi_n^2 = r^2(\beta_1^2 + \cdots + \beta_n^2) = r^2.$$

Therefore Eq. (4.1) becomes

$$t^{n-3}\mathscr{H}(x, t) = \frac{t^{n-3}}{\omega_n} \int\cdots\int_{|\xi|\leqslant t} \frac{g(x_1 + \xi_1, \ldots, x_n + \xi_n)}{|\,\xi\,|^{2\mu-2}\,(t^2 - |\,\xi\,|^2)^{1/2}}\, d\xi. \tag{4.2}$$

In (4.2) we make the change of variables

$$\xi_l = \chi_l t \qquad l = 1, \ldots, n,$$
$$d\xi = t^n\, d\chi_1 \cdots d\chi_n \equiv t^n\, d\chi.$$

Then Eq. (4.2) becomes

$$t^{n-3}\mathscr{H}(x, t) = \frac{t^{n-2}}{\omega_n} \int\cdots\int_{|\chi|\leqslant 1} \omega_n(|\,\chi\,|)\, g(x + \chi t)\, d\chi \tag{4.3}$$

where

$$g(x + \chi t) \equiv g(x_1 + \chi_1 t, \ldots, x_n + \chi_n t),$$
$$|\,\chi\,|^2 = \chi_1^2 + \cdots + \chi_n^2,$$
$$\omega_n(|\,\chi\,|) = |\,\chi\,|^{2-2\mu}(1 - |\,\chi\,|^2)^{-1/2}.$$

Since t does not occur in the limits of integration in (4.3) differentiation with respect to t is not difficult.

From (1.6) and (4.3) we obtain for $n = 2$

$$\psi(g; x, t) = \frac{t}{\omega_2} \int \cdots \int_{|x| \leqslant 1} \frac{g(x_1 + \chi_1 t, x_2 + \chi_2 t)}{(1 - \chi_1^2 - \chi_2^2)^{1/2}} \, d\chi. \tag{4.4}$$

In general for $n = 2\mu$, $\mu \geqslant 1$

$$\psi(g; x, t) = (t/\omega_n) \int \cdots \int_{|x| \leqslant 1} w_n(| \chi |) \, \hat{g}(x + \chi t) \, d\chi \tag{4.5}$$

where

$$\hat{g}(x + \chi t) = g(x + \chi t) + b^g_{\mu,1} t g_t(x + \chi t) + \cdots + b^g_{\mu,\mu-1} t^{\mu-1} \frac{\partial^{\mu-1}}{\partial t^{\mu-1}} g(x + \chi t) \tag{4.6}$$

where the $b^g_{\mu,j}$ are constants.

Now we consider cubature formulas for integral (4.5). If $g(\xi_1, \ldots, \xi_n)$ is a polynomial of degree m in $\xi_1, \ldots, \xi_n$ then $g(x + \chi t)$ and $\hat{g}(x + \chi t)$ are polynomials of degree m in $\chi_1, \ldots, \chi_n$. Since cubature formulas for (4.5) for $(x, t) \neq (0, t)$ are obtained by a linear transformation from formulas for $(0, t)$ we can assume that $(x, t) = (0, t)$.

THEOREM 2 *Assume that*

$$(\beta_{k1}, \ldots, \beta_{kn}), \qquad B_k, \qquad k = 1, \ldots, K \tag{4.7}$$

is a cubature formula of degree d for U_n. Assume that

$$r_j, \qquad A_j, \qquad j = 1, \ldots, J \tag{4.8}$$

are found so that the approximation

$$\frac{t}{\omega_n} \int \cdots \int_{|x| \leqslant 1} w_n(| \chi |) \, \hat{g}(\chi t) \, d\chi \simeq \sum_{j=1}^{J} \sum_{k=1}^{K} A_j B_k \, \hat{g}(r_j \beta_{k1} t, \ldots, r_j \beta_{kn} t) \tag{4.9}$$

is exact for the functions

$$\hat{g}(\xi_1, \ldots, \xi_n) = (\xi_1^2 + \cdots + \xi_n^2)^\nu, \qquad \nu = 0, 1, \ldots, [d/2]. \tag{4.10}$$

Then formula (4.9) *has degree d.*

The proof will be omitted; it is essentially the same as the proof of a similar result of Stroud [4, Theorem 2.8–2, p. 45].

The requirement that (4.9) be exact for the functions (4.10) means that the r_j, A_j must satisfy

$$\sum_{j=1}^{J}\sum_{k=1}^{K} A_j B_k[r_j{}^2\beta_{k1}^2 t^2 + \cdots + r_j{}^2\beta_{kn}^2 t^2]^\nu = I_\nu \qquad \nu = 0, 1,\ldots, [d/2],$$
$$I_\nu = \frac{t^{2\nu+1}}{\omega_n}\int_{|\chi|<1}\cdots\int w_n(|\chi|)(\chi_1{}^2 + \cdots + \chi_n{}^2)^\nu\, d\chi. \tag{4.11}$$

If we make the change of variables

$$\chi_l = r\beta_l, \qquad l = 1,\ldots, n, \qquad d\chi = r^{n-1}\, d\beta\, dr$$

integral I_ν becomes

$$I_\nu = \frac{t^{2\nu+1}}{\omega_n}\int_0^1 \frac{r^{n+2\nu-1}}{r^{n-2}(1-r^2)^{1/2}}\, dr \int_{U_n}\cdots\int d\beta = t^{2\nu+1}\int_0^1 \frac{r^{2\nu+1}}{(1-r^2)^{1/2}}\, dr.$$

Because

$$\sum_{k=1}^{K} B_k = 1 \qquad \text{<nd} \qquad \beta_{k1}^2 + \cdots + \beta_{kn}^2 = 1, \qquad k = 1,\ldots, K$$

Eqs. (4.11) simplify to

$$\sum_{j=1}^{J} A_j r_j^{2\nu} = (I_\nu/t^{2\nu}), \qquad \nu = 0, 1,\ldots, [d/2]. \tag{4.12}$$

We are interested in finding the r_j, A_j so that (4.12) is valid with J as small as possible. Note that the I_ν are independent of n so the same is true of the r_j, A_j. To simplify the discussion we assume d is an odd integer $d = 2d_0 - 1$. The problem of finding the r_j, A_j reduces to a problem of finding either a Gauss quadrature formula or a Radau formula.

In (4.12) we make the substitution

$$\tilde{r} = r^2, \qquad \tilde{r}_j = r_j{}^2, \qquad \tilde{A}_j t = A_j, \quad j = 1,\ldots, J.$$

Equations (4.12) become

$$\sum_{j=1}^{J} \tilde{A}_j \tilde{r}_j{}^\nu = I_\nu{}^* \qquad \nu = 0, 1,\ldots, [d/2],$$
$$I_0{}^* = 1, \qquad I_\nu{}^* = \frac{2\cdot 4\cdots(2\nu)}{3\cdot 5\cdots(2\nu+1)} \qquad \nu > 0.$$

If d_0 is even the $\tilde{r}_j$, $\tilde{A}_j$ will be a Gauss formula, with $J = d_0/2$, for the interval [0, 1] and weight function $(1-\tilde{r})^{-1/2}/2$. If d_0 is odd, we set $\tilde{r}_1 = 0$ and the $\tilde{r}_j$, $\tilde{A}_j$ will be a Radau formula with $J = (d_0 + 1)/2$.

TABLE I

Values of the r_j, A_j in Approximation (4.9)

d	J	r_j	$\tilde{A}_j = A_j/t$
3	1	0.816496580927726	1.000000000000000
5	2	0.000000000000000	0.166666666666667
		0.894427190999916	0.833333333333333
7	2	0.508374126853630	0.347854845137454
		0.940432288898543	0.652145154862546
9	3	0.000000000000000	0.066666666666667
		0.643964557508549	0.378474956297847
		0.958458649085161	0.554858377035486
11	3	0.361248674851186	0.171324492379170
		0.750201404456767	0.360761573048139
		0.971113218956983	0.467913934572691
13	4	0.000000000000000	0.035714285714286
		0.489968482125584	0.210704227143506
		0.806158108122710	0.341122692483504
		0.977851643852691	0.412458794658704
15	4	0.279004285823608	0.101228536290376
		0.604419162308214	0.222381034453374
		0.850773580999984	0.313706645877887
		0.983031907891343	0.362683783378362
17	5	0.000000000000000	0.022222222222222
		0.393010676358949	0.133305990851070
		0.673953393223206	0.224889342063126
		0.878400673012427	0.292042683679684
		0.986246858627551	0.327539761183897
19	5	0.226949495949278	0.066671344308688
		0.501662607349521	0.149451349150581
		0.733759251051551	0.219086362515982
		0.901203879456560	0.269266719309996
		0.988856122594578	0.295524224714753

In Table I we give values of the r_j and $\tilde{A}_j = A_j/t$ for $d = 3(2)19$. Computer subroutines WH2G and WH4G, in [5], approximate ψ for $n = 2$ and 4, respectively, using this method. In WH2G d may have one of the values $d = 7(4)31$. In WH4G d may have one of the values $d = 3(2)11$.

4.2. $n = 2\mu + 1$

For $n = 2\mu + 1$, $\mu \geqslant 1$, Eq. (1.7) is

$$\psi(g; x, t) = (t/\omega_n) \int_{U_n}\cdots\int g^*(x + \beta t)\, d\beta, \tag{4.13}$$

$$g^*(x + \beta t) = g(x + \beta t) + c^g_{\mu,1} t g_t(x + \beta t) + \cdots + c^g_{\mu,\mu-1} t^{\mu-1} \frac{\partial^{\mu-1}}{\partial t^{\mu-1}} g(x + \beta t), \tag{4.14}$$

TABLE II

An Example for $n = 2$. $u(x, y, t) = (x + y + t)^{5/2}$; $(x_0, y_0, t_0) = (4, 2, 3)$

Degree d	ϕ_1[WH2F]	ϕ_2[WH2F2; II=0]	ϕ_3[WH2F2; $II = 1$][a]	ψ_1[WH2G]
7	169.927183378256	169.926732877911	169.929251570901	117.261927109887
11	169.927364647192	169.927362437797	169.927369122044	117.261868104647
15	169.927360324192	169.927360301498	169.927360347843	117.261869785766
19	169.927360338294	169.927360337995	169.927360338460	117.261869779887
23	169.927360337623	169.927360337618	169.927360337621	117.261869780179
True value	169.927360337626	169.927360337626	169.927360337626	117.261869780178
	ζ_1[WNH2]	$\phi_1 + \psi_1 + \zeta_1$	$\phi_2 + \psi_1 + \zeta_1$	$\phi_3 + \psi_1 + \zeta_1$
7	−44.1892295595634	242.999880928580	242.999430428235	243.001949121224
11	−44.1892306977665	243.000002054073	242.999999844678	243.000006528925
15	−44.1892301173380	242.999999992620	242.999999969926	243.000000016271
19	−44.1892301179068	243.000000000274	242.999999999975	243.000000000440
23	−44.1892301178032	242.999999999998	242.999999999994	242.999999999997
True value	−44.1892301178034	243.000000000000	243.000000000000	243.000000000000

[a] The approximations ϕ_3 obtained with subroutine WH2F2 and $II = 1$ have degree $d + 2$.

TABLE III

An Example for $n = 3$. $u(x, y, z, t) = \ln(x + y + z + t)$, $(x_0, y_0, z_0, t_0) = (2, 2, 2, 3)$

Degree d	ϕ[WH3F]	ψ[WH3G]	ζ[WNH3]	$\phi + \psi + \zeta$
7	1.09916878111769	0.760084168319094	0.335551444302545	2.19480439373933
11	1.09894133978225	0.760188014056073	0.337948234902239	2.19707758874056
15	1.09853066812567	0.760385575582735	0.338271687104515	2.19718793081292
19	1.09862258158627	0.760340975677644	0.338261386193157	2.19722494345707
23	1.09861138031049	0.760346441244328		
27	1.09861233233823	0.760345974840453		
31	1.09861229094568	0.760345995180748		
True value	1.09861228866811	0.760345996300946	0.338266292357163	2.19722457733622

TABLE IV

Examples for $n = 4$. $\phi = \cos 2t \sin(x + y + z + w)$, $(x_0, y_0, z_0, w_0, t_0) = (1, 1, 1, 1, 1)$

Degree d	ϕ_1[WH4F]	ϕ_2[WH4F2]	
3	0.300172409268669	0.474869958515314	
5	0.308816007976183	0.314673037801367	
7	0.315017236581062	0.314894601835653	
9	0.314940652750427	0.314941706417080	
11	0.314940974168969	0.314940966425168	
True value	0.314940964313378	0.314940964313378	
	$\psi = t \cos x \cosh y \cos z \cosh w$, $(x_0, y_0, z_0, w_0, t_0) = (1, 1, 1, 1, 1)$		
	ψ_1[WH4G]	ψ_2[WH4G2; $II = 0$]	ψ_3[WH4G2; $II=1$]
3	0.675873868798821	0.669606158076402	0.675900669984835
5	0.695765821490395	0.695848819704875	0.695807279033372
7	0.695095023238752	0.695094835550887	0.695094905936746
9	0.695105864005589	0.695105864648501	0.695105864487604
11	0.695105754043557	0.695105754042633	0.695105754042838
True value	0.695105754805187	0.695105754805187	0.695105754805187

where the $c^g_{\mu,j}$ are constants. For $n = 3$ only the first term appears on the right side of Eq. (4.14).

Integral (4.13) can be approximated by a cubature formula for U_n. Subroutine WH3G approximates (4.13) using a Gauss product formula of degree d, where d has one of the values $4 = 7(4)31$.

5. Cubature Formulas for ϕ

5.1. $n = 2\mu$

From Eq. (1.10) and the results of Section 4.1 we find that

$$\phi(f; x, t) = (1/\omega_n) \int_{|\chi|<1}\cdots\int w_n(|\chi|) \hat{f}(x + \chi t)\, d\chi \tag{5.1}$$

where

$$\hat{f}(x + \chi t) = f(x + \chi t) + b^f_{\mu,1} t f_t(x + \chi t) + \cdots + b^f_{\mu,\mu} t^\mu \frac{\partial^\mu}{\partial t^\mu} f(x + \chi t)$$

where the $b^f_{\mu,j}$ are constants.

Assume that

$$\begin{array}{lll} (\beta_{kl}, \ldots, \beta_{kn}), & B_k, & k = 1, \ldots, K, \\ r_j, & A_j = \bar{A}_j t, & j = 1, \ldots, J, \end{array}$$

are as defined in Theorem 2. Then the approximation

$$(1/\omega_n) \int\cdots\int_{|x|<1} w_n(|\chi|) f(\chi t)\, d\chi \simeq \sum_{j=1}^{J} \sum_{k=1}^{K} \tilde{A}_j B_k f(r_j \beta_{k1} t, \ldots, r_j \beta_{kn} t)$$

is exact whenever f is a polynomial of degree $\leqslant d$.

Subroutines WH2F and WH4F in [5] use this method to approximate $\phi(f; x, t)$ for $n = 2$ and 4, respectively. In WH2F d may have the values $d = 7(4)31$; in WH4F d may have the values $d = 3(2)11$.

5.2. $n = 2\mu + 1$

From (1.7) and (1.10) we have

$$\phi(f; x, t) = (1/\omega_n) \int\cdots\int_{U_n} f^*(x + \beta t)\, d\beta \tag{5.2}$$

where

$$f^*(x + \beta t) = f(x + \beta t) + c^f_{\mu,1} t f_t(x + \beta t) + \cdots + c^f_{\mu,\mu} t^\mu \frac{\partial^\mu}{\partial t^\mu} f(x + \beta t)$$

where the $c^f_{\mu,j}$ are constants.

Subroutine WH3F approximates (5.2) by a Gauss product formula of degree d where d may have one of the values $d = 7(4)31$.

6. Other Approximations for ψ

For $n = 2$, 3 the approximations of Section 4 are linear combinations of values of g. For $n \geqslant 4$ those approximations also contain partial derivatives of g. In applications it may be inconvenient to use derivatives; therefore in this section we seek approximations for ψ which only contain values of g for $n \geqslant 4$. Again, because of the linearity of the approximations we can assume $(x, t) = (0, t)$.

Theorem 3. *Assume that*

$$(\beta_{kl}, \ldots, \beta_{kn}), \qquad B_k, \qquad k = 1, \ldots, K \tag{6.1}$$

is a cubature formula of degree d for U_n. Assume that

$$r_j, \qquad A_j, \qquad j = 1, \ldots, J \tag{6.2}$$

are found so that the approximation

$$\psi(g; 0, t) \simeq \sum_{j=1}^{J} \sum_{k=1}^{K} A_j B_k g(r_j \beta_{k1} t, \ldots, r_j \beta_{nk} t) \tag{6.3}$$

is exact for the functions

$$g(x_1, \dots, x_n) = (x_1^2 + \cdots + x_n^2)^\nu, \qquad \nu = 0, 1, \dots, [d/2]. \tag{6.4}$$

Then approximation (6.3) *is exact whenever g is a polynomial of degree* $\leqslant d$.

Proof. To show that (6.3) is exact for all polynomials g of degree $\leqslant d$ it suffices to show that (6.3) is exact whenever ψ is a wave polynomial which belongs to one of the sets $Z_{1,n}^g, \dots, Z_{d+1,n}^g$. This follows from Theorem 1(iii). Any such $\psi(g; x, t)$ has the form (3.2) where $1 \leqslant m \leqslant d + 1$; $m - s$ is an odd integer; $0 \leqslant s \leqslant m$; and $H_{s,n}$ is an arbitrary homogeneous harmonic polynomial of degree s. For such a ψ the corresponding g is given by expression (3.5).

Consider two cases. Case I, $s > 0$. In this case

$$H_{s,n}^h(0, \dots, 0) = 0.$$

Therefore

$$\psi(g; 0, t) = Q_{m,s,n}(0, t) = 0. \tag{6.5}$$

Next we compute the sum in (6.3) for this ψ. Using the fact that $\beta_{k1}^2 + \cdots + \beta_{kn}^2 = 1$, for all k, and the fact that

$$\sum_{k=1}^{K} B_k H_{s,n}^h(\beta_{k1}, \dots, \beta_{kn}) = 0$$

(see Eq. (2.3)) we find that the sum in (6.3) is

$$(-1)^{(m-s-1)/2} e_{m-s,1}^{\lambda} t^{m-1} \sum_{j=1}^{J} A_j r_j^{m-1} \sum_{k=1}^{K} B_k H_{s,n}^h(\beta_{k1}, \dots, \beta_{kn}) = 0.$$

Combined with (6.5), this shows that (6.3) is exact for all ψ in case $s > 0$; this is true regardless of the values chosen for the r_j, A_j.

Case II, $s = 0$. The wave polynomials which occur in this case are

$$\psi(g; x, t) = Q_{m,0,n}, \qquad m = 1, 3, \dots, M$$

where $M = 2[d/2] + 1$ is the largest odd integer $\leqslant d + 1$. The g which corresponds to these ψ are

$$(-1)^{(m-1)/2} e_{m,1}^{(n-1)/2} (x_1^2 + \cdots + x_n^2)^{(m-1)/2}, \qquad m = 0, 1, \dots, M. \tag{6.6}$$

But these functions are just constant multiples of the functions (6.4). Therefore if (6.3) is exact for the functions (6.4) it will also be exact for the functions (6.6). This completes the proof of Theorem 3.

Now we discuss the possibility of finding the r_j, A_j in Theorem 3. We assume $d = 2d_0 - 1$. For the function

$$g = (x_1^2 + \cdots + x_n^2)^\nu \tag{6.7}$$

the sum in (6.3) is

$$t^{2\nu} \sum_{j=1}^{J} A_j r_j^{2\nu}.$$

We consider the separate cases of n even and n odd.

6.1. $n = 2\mu$

From (1.6) the $\psi(g; 0, t)$ which corresponds to function (6.7) is

$$\psi_\nu \equiv \psi(g; 0, t) = \frac{t}{(\mu - 1)!} \left(\frac{\partial}{\partial t^2}\right)^{\mu-1} \left\{\frac{t^{2\mu-3}}{\omega_n} \int_0^t \frac{r}{(t^2 - r^2)^{1/2}} \int_{U_n}\cdots\int | \beta |^{2\nu} r^{2\nu} \, d\beta \, dr\right\}$$

$$= \frac{t}{(\mu - 1)!} \left(\frac{\partial}{\partial t^2}\right)^{\mu-1} \left\{t^{2(\mu+\nu-1)} \int_0^1 \frac{\tau^{2\nu+1}}{(1 - \tau^2)^{1/2}} \, d\tau\right\}.$$

Then

$$\psi_0 = t, \qquad \psi_\nu = \frac{(\mu + \nu - 1)!}{\nu! \, (\mu - 1)!} \frac{2 \cdot 4 \cdots (2\nu)}{3 \cdot 5 \cdots (2\nu + 1)} t^{2\nu+1} \qquad \nu > 0. \tag{6.8}$$

Therefore the r_j, A_j must satisfy

$$\sum_{j=1}^{J} A_j r_j^{2\nu} = \psi_\nu / t^{2\nu}, \qquad \nu = 0, 1, \ldots, d_0 - 1. \tag{6.9}$$

We are interested in satisfying (6.9) with J as small as possible, $J = [(d_0 + 1)/2]$. We summarize our results.

$d = 3$, $J = 1$:

$$r_1 = (2\mu/3)^{1/2} \qquad A_1 = t;$$

$d = 5$, $J = 2$:

$$r_1 = 0, \qquad r_2 = \left(\frac{2\mu + 2}{5}\right)^{1/2}, \qquad A_1 = \left(\frac{3 - 2\mu}{3\mu + 3}\right) t, \qquad A_2 = \left(\frac{5\mu}{3\mu + 3}\right) t;$$

$d = 7$, $J = 2$:

$$r_1^2, r_2^2 = \frac{10\mu + 10 \pm 2(5(\mu + 1)(5 - 2\mu))^{1/2}}{35};$$

$d = 9$, $J = 3$:

$$r_1 = 0, \qquad r_2^2, r_3^2 = \frac{14\mu + 28 \pm 2(7(\mu + 2)(5 - 2\mu))^{1/2}}{63}.$$

With parameter $II = 0$, subroutine WH4G2 in [5] uses the above approximations for $n = 4$ for $d = 3(2)11$. Each of these approximations uses some values of $g(x + \chi t)$ at points outside the region $| \chi | \leqslant 1$. For $\mu \geqslant 3$, $d \geqslant 7$, these approximations cannot be used since some of the r_j are complex.

Now we consider the possibility of finding approximations (6.3) with all the $r_j \leqslant 1$. We assume $r_J = 1$ with J as small as possible, $J = [(d_0 + 2)/2]$.

$d = 3$, $J = 2$:

$$r_1 = 0, \qquad r_2 = 1, \qquad A_1 = \left(\frac{3 - 2\mu}{3}\right) t, \qquad A_2 = \frac{2\mu t}{3};$$

$d = 5$, $J = 2$:

$$r_1 = \left(\frac{2\mu}{5}\right)^{1/2}, \qquad r_2 = 1, \qquad A_1 = \left(\frac{15 - 10\mu}{15 - 6\mu}\right) t, \qquad A_2 = \left(\frac{4\mu}{15 - 6\mu}\right) t;$$

$d = 7$, $J = 3$:

$$r_1 = 0, \qquad r_2 = \left(\frac{2\mu + 2}{7}\right)^{1/2} \qquad A_1 = \left(\frac{4\mu^2 - 16\mu + 15}{15\mu + 15}\right)$$

$$r_3 = 1, \qquad A_2 = \frac{49\mu(3 - 2\mu)\, t}{15(\mu + 1)(5 - 2\mu)} \qquad A_3 = \frac{8\mu(\mu + 1)\, t}{15(5 - 2\mu)};$$

$d = 9$, $J = 3$:

$$r_3 = 1, \qquad r_1^2, r_2^2 = \frac{14\mu + 14 \pm 2(7(\mu + 1)(7 - 2\mu))^{1/2}}{63}.$$

With parameter $II = 1$ subroutine WH4G2 uses approximation (6.3) with the r_j, A_j we have just described for $n = 4$, $d = 3(2)11$. In these approximations all the points are inside the region $|\chi| \leqslant 1$. It is not true that all the r_j are $\leqslant 1$ for $\mu > 2$. For $\mu = 3$, $r_{J-1} > 1$ for $d \geqslant 5$ and for $\mu > 3$ some of the r_j are complex for $d \geqslant 7$.

6.2. $n = 2\mu + 1$

From (1.7) the $\psi(g; 0, t)$ which corresponds to function (6.7) is

$$\psi_\nu \equiv \psi(g; 0, t) = \frac{\pi^{1/2}}{2\Gamma(n/2)} \left(\frac{\partial}{\partial t^2}\right)^{\mu-1} \left\{\frac{t^{2\mu-1}}{\omega_n} \int_{U_n}\!\!\cdots\!\int |\beta|^{2\nu} t^{2\nu}\, d\beta\right\}$$

$$= \frac{\pi^{1/2}}{2\Gamma(n/2)} \left(\frac{\partial}{\partial t^2}\right)^{\mu-1} t^{2\mu+2\nu-1}.$$

The r_j, A_j must satisfy (6.9) with

$$\psi_0 = t, \qquad \psi_\nu = \frac{(2\nu + 3)(2\nu + 5) \cdots (2\nu + 2\mu - 1)}{3 \cdot 5 \cdots (2\mu - 1)} t^{2\nu+1}, \qquad \nu > 0. \quad (6.10)$$

For $J = [(d_0 + 1)/2]$ the results are as follows.

$d = 3$, $J = 1$:

$$r_1 = \left(\frac{2\mu + 1}{3}\right)^{1/2} \qquad A_1 = t;$$

$d = 5$, $J = 2$:

$$r_1 = 0, \qquad r_2 = \left(\frac{2\mu + 3}{5}\right)^{1/2}, \qquad A_1 = \frac{4(1-\mu)\,t}{3(2\mu+3)}, \qquad A_2 = \frac{5(2\mu+1)\,t}{3(2\mu+3)}.$$

For $\mu \geqslant 2$, $d \geqslant 7$, the r_j, A_j do not exist.
We also investigated the r_j, A_j with $r_J = 1$, $J = [(d_0 + 2)/2]$.

$d = 3$, $J = 2$:

$$r_1 = 0, \qquad r_2 = 1, \qquad A_1 = \frac{2(1-\mu)\,t}{3}, \qquad A_2 = \frac{(2\mu+1)\,t}{3};$$

$d = 5$, $J = 2$:

$$r_1 = \left(\frac{2\mu + 1}{5}\right)^{1/2}, \qquad r_2 = 1, \qquad A_1 = \frac{5(1-\mu)\,t}{3(2-\mu)}, \qquad A_2 = \frac{(2\mu+1)\,t}{3(2-\mu)}.$$

Again for $\mu \geqslant 2$, $d \geqslant 7$, the r_j, A_j do not exist.

7. Other Approximations for ϕ

The approximations of Section 5 use partial derivatives of f for all $n \geqslant 2$. Here we seek approximations for $\phi(f; 0, t)$ which only use values of f.

Theorem 4. *Assume that*

$$(\beta_{kl}, \ldots, \beta_{kn}), \qquad B_k, \qquad k = 1, \ldots, K \tag{7.1}$$

is a cubature formula of degree d for U_n. Assume that

$$r_j, \qquad A_j, \qquad j = 1, \ldots, J \tag{7.2}$$

are found so that the approximation

$$\phi(f; 0, t) \simeq \sum_{j=1}^{J} \sum_{k=1}^{K} A_j B_k f(r_j \beta_{k1} t, \ldots, r_j \beta_{kn} t) \tag{7.3}$$

is exact for the functions

$$f(x_1, \ldots, x) = (x_1^2 + \cdots + x_n^2)^\nu, \qquad \nu = 0, 1, \ldots, [d/2]. \tag{7.4}$$

Then approximation (7.3) *is exact whenever f is a polynomial of degree $\leqslant d$.*

The proof of Theorem 4 will be omitted; it is analogous to the proof of Theorem 3. The requirement that approximation (7.3) be exact for the functions (7.4) leads to the equations

$$t^{2\nu} \sum_{j=1}^{J} A_j r_j^{2\nu} = (\partial/\partial t)\, \psi_\nu\,, \qquad \nu = 0, 1,..., [d/2]. \tag{7.5}$$

We consider the cases of n even and n odd.

7.1. $n = 2\mu$

In this case the ψ_ν in (7.5) are given by (6.8). Then Eqs. (7.5) become

$$\sum_{j=1}^{J} A_j r_j^{2\nu} = \phi_\nu{}^*, \qquad \nu = 0, 1,..., [d/2]$$

where

$$\phi_0{}^* = 1, \qquad \phi_\nu{}^* = \frac{(\mu + \nu - 1)!}{\nu!\,(\mu - 1)!}\,\frac{2 \cdot 4 \cdots (2\nu)}{3 \cdot 5 \cdots (2\nu - 1)}\,, \qquad \nu > 0.$$

The results for $J = [(d_0 + 1)/2]$ are as follows.

$d = 3$, $J = 1$:

$$r_1 = (2\mu)^{1/2} \qquad A_1 = 1;$$

$d = 5$, $J = 2$:

$$r_1 = 0, \qquad r_2 = \left(\frac{2\mu + 2}{3}\right)^{1/2}, \qquad A_1 = \frac{1 - 2\mu}{\mu + 1}\,, \qquad A_2 = \frac{3\mu}{\mu + 1};$$

$d = 7$, $J = 2$:

$$r_1{}^2, r_2{}^2 = \frac{6\mu + 6 \pm 2(3(\mu + 1)(3 - 2\mu))^{1/2}}{15};$$

$d = 9$, $J = 3$:

$$r_1 = 0, \qquad r_2{}^2, r_3{}^2 = \frac{10\mu + 20 \pm (5(\mu + 2)(3 - 2\mu))^{1/2}}{35}.$$

With parameter $II = 0$ subroutine WH2F2 uses the above approximation for $n = 2$ and $d = 7(4)31$. For $n \geqslant 4$, $d \geqslant 7$ these approximations cannot be used because some of the r_j are complex.

The results for $r_J = 1$, $J = [(d_0 + 2)/2]$ are as follows.

$d = 3$, $J = 2$:

$$r_1 = 0, \qquad r_2 = 1, \qquad A_1 = 1 - 2\mu, \qquad A_2 = 2\mu;$$

$d = 5,\ J = 2$:

$$r_1 = \left(\frac{2\mu}{3}\right)^{1/2}, \qquad r_2 = 1, \qquad A_1 = \frac{3 - 6\mu}{3 - 2\mu}, \qquad A_2 = \frac{4\mu}{3 - 2\mu};$$

$d = 7,\ J = 3$:

$$r_1 = 0, \qquad r_2 = \left(\frac{2\mu + 2}{5}\right)^{1/2} \qquad A_1 = \frac{(2\mu - 1)(2\mu - 3)}{3(\mu + 1)}$$

$$r_3 = 1, \qquad A_2 = \frac{25\mu(1 - 2\mu)}{3(\mu + 1)(3 - 2\mu)} \qquad A_3 = \frac{8\mu(\mu + 1)}{3(3 - 2\mu)};$$

$d = 9,\ J = 3$:

$$r_3 = 1, \qquad r_1^2, r_2^2 = \frac{10\mu + 10 \pm 2(5(\mu + 1)(5 - 2\mu))^{1/2}}{35}.$$

In general let $r_1, r_2, \ldots, r_{J-1}, r_J = 1$ denote the r_j for a formula of degree $d = 2d_0 - 1$ which we have just described. Then $r_1, \ldots, r_{J-1}$ coincide with the r_j in the formula of degree $d = 2d_0 - 3$ of Section 6.1 with $J = [(d_0 + 1)/2]$.

With parameter $II = 1$, subroutine WH2F2 uses approximation (7.3) with the above described r_j, A_j for $n = 2$ and $d = 7(4)31$. Subroutine WH4F2 uses this approximation for $n = 4$ and $d = 3(2)11$.

7.2. $n = 2\mu + 1$

In this case ψ_ν in (7.5) is given by (6.10). Equations (7.5) become

$$\sum_{j=1}^{J} A_j r_j^{2\nu} = \phi_\nu^*, \qquad \nu = 0, 1, \ldots, [d/2],$$

$$\phi_\nu^* = \frac{(2\nu + 1)(2\nu + 3) \cdots (2\nu + 2\mu - 1)}{1 \cdot 3 \cdots (2\mu - 1)} \qquad \nu \geqslant 0.$$

The results for $J = [d_0 + 1)/2]$ are as follows.

$d = 3,\ J = 1$:

$$r_1 = (2\mu + 1)^{1/2}, \qquad A_1 = 1;$$

$d = 5,\ J = 2$:

$$r_1 = 0, \qquad r_2 = \left(\frac{2\mu + 3}{3}\right)^{1/2}, \qquad A_1 = \frac{-4\mu}{2\mu + 3}; \qquad A_2 = \frac{3(2\mu + 1)}{2\mu + 3}$$

The r_j, A_j do not exist for $d \geqslant 7$ for all $\mu \geqslant 1$.

For $J = [(d_0 + 2)/2]$, $r_J = 1$, the results are as follows.

$d = 3, J = 2$:

$$r_1 = 0, \qquad r_2 = 1, \qquad A_1 = -2\mu, \qquad A_2 = 2\mu + 1;$$

$d = 5, J = 2$:

$$r_1 = \left(\frac{2\mu+1}{3}\right)^{1/2}, \qquad r_2 = 1, \qquad A_1 = \frac{3\mu}{\mu - 1}, \qquad A_2 = \frac{2\mu+1}{1-\mu}.$$

Again the r_j, A_j do not exist for $d \geqslant 7$ for all $\mu \geqslant 1$.

8. Formulas for ζ

From representation (1.11) we can obtain formulas for ζ from formulas for ψ. Here we only explicitly consider formulas for ζ which are obtained from the formulas for ψ given in Section 4. For a given $F(x, t)$ we define $\hat{F}(x, t)$ and $F^*(x, t)$ by Eq. (4.6) and (4.14), respectively.

THEOREM 5. *Assume that*

$$\int_0^1 \eta P(\eta)\, d\eta \simeq \sum_{l=1}^{L} C_l P(\eta_l) \tag{8.1}$$

is a quadrature formula of degree d. Assume that

$$(\beta_{kl}, \ldots, \beta_{kn}), \qquad B_k, \qquad k = 1, \ldots, K, \tag{8.2}$$

$$r_j, \qquad A_j = \tilde{A}_j t, \qquad j = 1, \ldots, J, \tag{8.3}$$

are defined as in Theorem 2. Then the approximations

$$\zeta(F; x, t) \simeq t^2 \sum_{j=1}^{J} \sum_{k=1}^{K} \sum_{l=1}^{L} \tilde{A}_j B_k C_l \hat{F}(x_1 + r_j\beta_{k1}\eta_l t, \ldots, x_n + r_j\beta_{kn}\eta_l t, (1 - \eta_l)\, t) \quad \text{for } n = 2\mu, \tag{8.4}$$

$$\zeta(F; x, t) \simeq t^2 \sum_{k=1}^{K} \sum_{l=1}^{L} B_k C_l F^*(x_1 + r_j\beta_{k1}\eta_l t, \ldots, x_n + r_j\beta_{kn}\eta_l t, (1 - \eta_l)\, t) \quad \text{for } n = 2\mu + 1 \tag{8.5}$$

have degree d.

Proof. We give the proof for approximation (8.4); the proof for (8.5) is analogous. Without loss of generality we can assume $(x, t) = (0, t)$. Then it suffices to prove that, for $x = 0$, the sum in (8.4) equals the integral

$$\int_0^t \psi(F; 0, t - \tau; \tau)\, d\tau = (1/\omega_n) \int_0^t \int \cdots \int_{|\chi| \leqslant 1} w_n(|\chi|)(t - \tau)\, \hat{F}(\chi(t - \tau), \tau)\, d\chi\, d\tau \tag{8.6}$$

when $\hat{F}$ is an arbitrary monomial

$$\hat{F}(\xi_1, \ldots, \xi_n, \tau) = \xi_1^{\alpha_1} \xi_2^{\alpha_2} \cdots \xi_n^{\alpha_n} \tau^{\sigma},$$
$$\alpha_1 + \cdots + \alpha_n + \sigma = s, \qquad 0 \leqslant s \leqslant d, \tag{8.7}$$

In (8.6) we make the change of variable $t - \tau = \eta t$. Then this integral becomes

$$(t^2/\omega_n) \int_0^1 \underset{|x| \leqslant 1}{\int \cdots \int} w_n(|\chi|)\, \eta \hat{F}(\chi\eta t, (1-\eta)\, t)\, d\chi\, d\eta. \tag{8.8}$$

For monomial (8.7) integral (8.8) is

$$\frac{t^{s+2}}{\omega_n} \int_0^1 \eta^{\alpha_1 + \cdots + \alpha_n + 1} (1-\eta)^{\sigma}\, d\eta \underset{|x| \leqslant 1}{\int \cdots \int} w_n(|\chi|)\, \chi_1^{\alpha_1} \cdots \chi_n^{\alpha_n}\, d\chi. \tag{8.9}$$

For monomial (8.7) and $x = 0$, the sum in (8.4) is found to be

$$t^{s+2} \sum_{l=1}^{L} C_l \eta_l^{\alpha_1 + \cdots + \alpha_n} (1 - \eta_l)^{\sigma} \sum_{j=1}^{J} \sum_{k=1}^{K} \tilde{A}_j B_k (r_j \beta_{k1})^{\alpha_1} \cdots (r_j \beta_{kn})^{\alpha_n}. \tag{8.10}$$

By the assumed properties of formulas (8.1)–(8.3) the expressions (8.9) and (8.10) are seen to be equal. Since (8.1) has degree d it follows that (8.4) is not exact for $\hat{F} = \tau^{d+1}$. This completes the proof.

Subroutines WNH2 and WNH3 in [5] use (8.4) and (8.5), respectively, to approximate $\zeta(x, t)$ for $n = 2$ and $n = 3$ for $d = 7(4)31$. Formula (8.1) is taken to be a Gauss formula.

9. Examples

Here we give the results of numerical examples of (1.1), in each of the dimensions $n = 2$, 3, and 4. The results are given in Tables 2, 3, and 4, respectively.

For $n = 2$, we solved (1.1) at $(x_0, y_0, t_0) = (4, 2, 3)$ with

$$\begin{aligned} F(x, y, t) &= -(15/4)(x + y + t)^{1/2}, \\ f(x, y) &= (x + y)^{5/2}, \\ g(x, y) &= (5/2)(x + y)^{3/2}. \end{aligned}$$

The true solutions are

$$\begin{aligned} \phi(x, y, t) &= \tfrac{1}{2}[(x + y + t2^{1/2})^{5/2} + (x + y - t2^{1/2})^{5/2}], \\ \psi(x, y, t) &= (2^{1/2}/4)[(x + y + t2^{1/2})^{5/2} - (x + y - t2^{1/2})^{5/2}], \\ u(x, y, t) &= (x + y + t)^{5/2} \qquad \zeta = u - \phi - \psi. \end{aligned}$$

We used the following subroutines listed in [5].

(i) For problem (1.2), subroutine WH2F, which uses the formulas of Section 5.1; this requires the function $f(x, y)$ and also the function

$$tf_t(x + \chi_1 t, y + \chi_2 t) = [(x - x_0)(\partial/\partial x) + (y - y_0)(\partial/\partial y)]\, f(x, y).$$

(ii) For problem (1.2), subroutine WH2F2 (with parameter $II = 0$), which uses formulas of Section 7.1; this requires only $f(x, y)$ but has some points outside the region of integration.

(iii) For problem (1.2), subroutine WH2F2 (with parameter $II = 1$), which uses formulas of Section 7.1; this requires only $f(x, y)$ and has all the points inside the region.

(iv) For problem (1.3), subroutine WH2G, which uses the formulas of Section 4.1.

(v) For problem (1.4), subroutine WNH2, which uses formulas of Section 8.

For $n = 3$ we solved (1.1) at $(x_0, y_0, z_0, t_0) = (2, 2, 2, 3)$ with

$$\begin{aligned} F(x, y, z, t) &= 2(x + y + z + t)^{-2}, \\ f(x, y, z) &= \ln(x + y + z), \\ g(x, y, z) &= (x + y + z)^{-1}. \end{aligned}$$

The true solutions are

$$\begin{aligned} \phi(x, y, z, t) &= \tfrac{1}{2} \ln[(x + y + z)^2 - 3t^2], \\ \psi(x, y, z, t) &= (3^{1/2}/6)[\ln(x + y + z + t3^{1/2}) - \ln(x + y + z - t3^{1/2})], \\ u(x, y, z, t) &= \ln(x + y + z + t) \qquad \zeta = u - \phi - \psi. \end{aligned}$$

We used the following subroutines from [5].

(vi) For problem (1.2), subroutine WH3F, which uses the formulas of Section 5.2

(vii) For problem (1.3), subroutine WH3G, which uses formulas of Section 4.2.

(viii) For problem (1.4), subroutine WNH 3, which uses the formulas of Section 8.

For $n = 4$ we have not provided a subroutine for (1.4); therefore we are restricted to the homogeneous problems (1.2) and (1.3).

We solved problem (1.2) at (1, 1, 1, 1, 1) with $f = \sin(x + y + z + w)$; the true solution is $\phi = f \cos 2t$. We used the following subroutines from [5].

(ix) WH4F, which uses formulas of Section 5.1; this requires $f(x_1, \dots, x_4)$, $tf_t(x_1 + \chi_1 t, \dots, x_4 + \chi_4 t)$ and

$$t^2 f_{tt}(x_1 + \chi_1 t, \dots, x_4 + \chi_4 t) = \sum_{j=1}^{4} \sum_{k=}^{4} (x_j - x_{0j})(x_k - x_{0k}) \frac{\partial^2}{\partial x_j\, \partial x_k} f(x_1, \dots, x_4).$$

(x) WH4F2, which uses formulas of Section 7.1; this requires only f.

We solved problem (1.3) at (1, 1, 1, 1, 1) with

$$g = \cos x \cosh y \cos z \cosh w.$$

The true solution is $\psi = tg$. We used the following subroutines.

(xi) WH4G, which uses formulas of Section 4.1; this requires $g(x, y, z, w)$ and $tg \cdot (x + \chi_1 t, \ldots, w + \chi_4 t)$.

(xii) WH4G2 (with parameter $II = 0$) which uses formulas of Section 6.1; this requires only g but has some points outside the region.

(xiii) WH4G2 (with parameter $II = 1$) which uses formulas of Section 6.1; this requires only g and has all points inside the region.

The error which is obtained in using the above formulas to solve problems (1.2)–(1.4) depends, to a considerable extent, on the smoothness of the functions f, g, and F. One should not assume that the error which is obtained in any particular problem will be of the same order of magnitude as that achieved in the above examples.

ACKNOWLEDGMENTS

The programs described above, and listed in [5], were written in Fortran double precision (which carries about 16 significant digits) for the AMDAHL 470V/6 computer at the Data Processing Center of Texas A & M University. This computer is compatible with the IBM 360/65. The constants in the integration formulas in these subroutines were computed in quadruple precision (which carries about 35 significant digits) and are given accurate to 15 decimals in the subroutines. We wish to thank Dr. Chuck Adams for assistance in using the quadruple precision Fortran compiler. We also wish to thank one of the referees for suggestions about the presentation of the paper and for providing true solutions for some of the examples.

REFERENCES

1. R. COURANT AND D. HILBERT, "Methods of Mathematical Physics," Vol. 2, Interscience, New York, 1962.
2. A. ERDÉLYI, Ed., "Higher Transcendental Functions," Vol. 2, McGraw–Hill, New York, 1953.
3. J. N. LYNESS AND D. JESPERSON, Moderate degree symmetric quadrature rules for the triangle, *J. Inst. Math. Appl.* **15** (1975), 19–32.
4. A. H. STROUD, "Approximate Calculation of Multiple Integrals," Prentice-Hall, Englewood Cliffs, N.J., 1971.
5. A. H. STROUD, "Computer Subroutines for Solving the Wave Equation in n Space Dimensions for $n = 2, 3, 4$," unpublished manuscript.
6. H. F. WEINBERGER, "A First Course in Partial Differential Equations," Blaisdell, Waltham, Mass., 1965.
7. G. E. FORSYTHE AND W. R. WASOW, "Finite-Difference Methods for Partial Differential Equations," Wiley, New York, 1960.
8. A. R. MITCHELL, "Computational Methods in Partial Differential Equations," Wiley, New York, 1969.